Homework 2: More Vector Analysis

Due: Wednesday, September 6^{th}

Important: You may not use MATHEMATICA or similar software on this homework. In fact, on most homework assignments you should assume that MATHEMATICA is not allowed. It is really important that you learn how to handle the math on these assignments without resorting to computers. There is a lot of understanding to be uncovered in working out the details for yourself.

Problem 1: Why We Love Conservative Forces

An object is moving in the x-y plane along the path $y = x^2 + x - 3$. It experiences a force

$$\vec{F}(x, y) = \frac{2\,\alpha\,x}{(x^2 + y^2)^2}\,\hat{x} + \frac{2\,\alpha\,y}{(x^2 + y^2)^2}\,\hat{y}\ , \tag{1}$$

where α is a constant. What is the work performed by this force as the object moves from the point $(2, 3)$ to the point $(0, -3)$?

> HINT: The work is the integral of $\vec{F} \cdot d\vec{\ell}$ along the path, from the start point to the end point. It will be messy if you try to do this directly. It can be done, but there is a *much* easier way to get the answer. Think about the title of the problem.

Problem 2: Divergence Theorem

The components of a vector function $\vec{A}$ are given in cylindrical polar coordinates $\{s, \phi, z\}$ as

$$\vec{A} = \frac{s^2}{L}\,\sin\left(\frac{\pi\,z}{L}\right)\,\hat{s} + s\,\cos\left(\frac{\pi\,z}{L}\right)\,\hat{z}\ , \tag{2}$$

where L is a constant with units of length.

(a) Use a surface integral to calculate the flux of $\vec{A}$ through the closed surface bounding the region $2\,L \le s \le 4\,L$, $0 \le z \le L$, $0 \le \phi < 2\pi$.

(b) Evaluate the same integral using the divergence theorem.

Problem 3: Stokes' Theorem

The components of a vector function $\vec{V}$ are given in Cartesian coordinates as

$$\vec{V} = \frac{x\,y}{a^2}\,\hat{x} + \left(\frac{x^2}{2\,a^2} + \frac{y\,z}{2\,a^2}\right)\hat{y} + \left(\frac{y^2}{4\,a^2} - \frac{2\,z}{a}\right)\hat{z}\ , \tag{3}$$

where a is a constant. Use Stokes' theorem to calculate the integral

$$\oint_P \vec{V} \cdot \mathrm{d}\vec{\ell} \tag{4}$$

where P is the rectangle $ABCD$ with corners $A = (-1, 1, 0)$, $B = (4, 1, 0)$, $C = (4, 4, 0)$, and $D = (-1, 4, 0)$.

Problem 4: The Three-Dimensional Delta Function

Perform the following integrals involving the three-dimensional delta function:

(a)

$$\int_{\substack{\text{all}\\\text{space}}} d\tau \, (r^2 - 5\,\vec{r}\cdot\vec{a} + 2\,a^2)\;\delta^3(\vec{r}-\vec{a}\,)\ ,$$

where $\vec{r}$ is the position vector with magnitude r, and $\vec{a}$ is a constant vector with magnitude a.

(b)

$$\int_{\mathcal{V}} d\tau \, \left|\vec{b}-\vec{r}\,\right|^2 \, \delta^3(5\,\vec{r}\,)\ ,$$

where $\mathcal{V}$ is a cube with sides of length 10, centered on the origin, and $\vec{b} = 7\,\hat{x} - 24\,\hat{z}$.

Problem 5: The Helmholtz Theorem

Consider the following vector functions:

$$\vec{F}_1 = y\,\hat{x} + z\,\hat{y} + 0\,\hat{z} \tag{5}$$

$$\vec{F}_2 = x^2\,\hat{x} - y^2\,\hat{y} + z^2\,\hat{z} \tag{6}$$

$$\vec{F}_3 = -y\,z\,\hat{x} - z\,x\,\hat{y} - x\,y\,\hat{z} \tag{7}$$

(a) Calculate the divergence and curl of $\vec{F}_1$ and $\vec{F}_2$.

(b) Which one of $\vec{F}_1$ and $\vec{F}_2$ can be written as the gradient of a scalar function? Find any scalar potential that does the job.

(c) Which one of $\vec{F}_1$ and $\vec{F}_2$ can be written as the curl of a vector function? Find any suitable vector potential.

(d) Show that $\vec{F}_3$ can be written both as the gradient of a scalar function and as the curl of a vector function.

(e) Find a scalar potential and a vector potential for $\vec{F}_3$.